1 Convex Sets [Yi, 12 points]

1.1
A slab, i.e., a set of the form \( \{ x \in \mathbb{R}^n | \alpha \leq a^T x \leq \beta \} \).

★ Solution: A slab is convex, since it is the intersection of two half-spaces \( \{ x \in \mathbb{R}^n | a^T x \geq \alpha \} \) and \( \{ x \in \mathbb{R}^n | a^T x \leq \beta \} \) and each half-space is a convex set.

1.2
The set of points closer to a given point than a given set, i.e.,
\[
\{ x | \| x - x_0 \|_2 \leq \| x - y \|_2 \text{ for all } y \in S \}
\]
where \( S \subseteq \mathbb{R}^n \)

★ Solution: This set is convex. For each \( y_0 \in S \), the set
\[
\{ x | \| x - x_0 \|_2 \leq \| x - y_0 \|_2 \}
\]
is a half-space and thus is a convex set. Then, the set
\[
\{ x | \| x - x_0 \|_2 \leq \| x - y \|_2 \text{ for all } y \in S \} = \bigcap_{y_0 \in S} \{ x | \| x - x_0 \|_2 \leq \| x - y_0 \|_2 \}
\]
is the intersection of convex sets, which is also convex.

1.3
The set \( \{ x | x + S_2 \subseteq S_1 \} \), where \( S_1, S_2 \subseteq \mathbb{R}^n \) with \( S_1 \) convex. Note that \( x + S_2 = \{ y | y = x + z, z \in S_2 \} \).

★ Solution: This set is convex. For any two points \( x_1 \) and \( x_2 \) in this set, by definition, we have
\[
x_1 + z \in S_1, \forall z \in S_2 \\
x_2 + z \in S_1, \forall z \in S_2
\]
To prove that this set is convex, consider any \( \lambda \in [0, 1] \), we have the following:
\[
[\lambda x_1 + (1 - \lambda)x_2] + z = \lambda(x_1 + z) + (1 - \lambda)(x_2 + z) \in S_1, \forall z \in S_2
\]
because for any \( z \in S_2 \) we have \( (x_1 + z), (x_1 + z) \) are in \( S_1 \) and \( S_1 \) is a convex set.
1.4

The set of points whose distance to \( a \) does not exceed a fixed fraction \( \theta \) of the distance to \( b \), i.e.,

\[
\{ x \mid \| x - a \|_2 \leq \theta \| x - b \|_2 \}
\]

(2)

where \( a \neq b \), \( 0 \leq \theta \leq 1 \), and \( a, b, x \in \mathbb{R}^n \).

★ Solution: This set is convex. For \( \theta = 1 \), the set is a halfspace and thus convex. Now consider \( 0 \leq \theta < 1 \):

\[
\{ x \mid \| x - a \|_2 \leq \theta \| x - b \|_2 \} = \{ x \mid \| x - a \|_2^2 \leq \theta^2 \| x - b \|_2^2 \} = \{ x |(x - a)^T(x - a) \leq \theta^2(x - b)^T(x - b) \} = \{ x |x^T x - 2x^T a + a^T a \leq \theta^2 x^T x - 2\theta^2 x^T b + \theta^2 b^T b \} = \{ x |(1 - \theta^2)x^T x - 2\theta^2 x^T a + \theta^2 b^T b \leq 0 \} = \{ x |x^T x - 2x^T \frac{(a - \theta^2 b)}{1 - \theta^2} + \frac{a^T a - \theta^2 b^T b}{1 - \theta^2} \leq 0 \}
\]

Now, you can either argue that this is the sublevel set of a convex function (e.g., the second-order derivative of the function is positive-definite), or if you are not familiar with convex functions, you can go ahead to derive:

\[
\{ x |x^T x - 2x^T \frac{(a - \theta^2 b)}{1 - \theta^2} + \frac{a^T a - \theta^2 b^T b}{1 - \theta^2} \leq 0 \} = \{ x |x - \frac{(a - \theta^2 b)}{1 - \theta^2} \}^T \{ x - \frac{(a - \theta^2 b)}{1 - \theta^2} \} \leq \frac{\theta^2(a - b)^T(a - b)}{(1 - \theta^2)^2} \}
\]

and argue that this is an ellipsoid.

2 Geometry of LPs [Yi, 10 points]

This question is designed to help you visualize the geometry of linear programming. Recall that if there are \( n \) variables in an LP, and there are \( m \) (linearly independent) equality constraints, then the solution lies in a \( n - m \) dimensional subspace. Consider the following linear program:

\[
\begin{aligned}
\text{min} & \quad 3x_1 + x_2 + x_3 + x_4 \\
\text{subject to} & \quad 3x_1 - x_2 - 7x_3 + x_4 = -3 \\
& \quad 3x_1 - 4x_3 + x_4 = -3 \\
& \quad x_3 \leq 3 \\
& \quad x_i \geq 0 \quad \forall i
\end{aligned}
\]

2.1

[5 pts] Draw the feasible set of solutions for the above LP. Enumerate the coordinates of the extreme vertice(s) of the feasible set (Hint: first transform to the inequality form of LP).

★ Solution: Since there are 4 variables and 2 equality constraints in the problem, the solution space is a 2 dimensional affine subspace. We can thus transform this problem into an equivalent optimization problem that involves only 2 variables. Let us choose \( \{x_1, x_3\} \), and eliminate \( x_2 \) and \( x_4 \):

\[
\begin{aligned}
3x_1 - 4x_3 + x_4 &= -1 \\
\implies x_4 &= -3x_1 + 4x_3 - 1 \\
3x_1 - x_2 - 7x_3 + x_4 &= -3 \\
\implies x_2 &= 3x_1 - 7x_3 + x_4 + 3 \\
\implies x_2 &= -3x_3 + 2
\end{aligned}
\]

(3)
By expressing $x_2$ and $x_4$ using $x_1$ and $x_3$, we have the following LP (equivalent to the original optimization problem):

$$\begin{align*}
\min & \quad 2x_3 + 1 \\
\text{subject to} & \quad 3x_1 - 4x_3 \leq -1 \\
& \quad x_3 \leq \frac{5}{9} \\
& \quad x_3 \leq 3 \\
& \quad x_1, x_3 \geq 0
\end{align*}$$

Note that the constraint $x_3 \leq 3$ is redundant given the constraint $x_3 \leq \frac{5}{9}$.

Since this optimization problem involves only 2 variables, we can visualize it as shown in Figure 1\(^1\).

![Figure 1: Visualizing the linear program](image)

From the diagram, the feasible region is shown with shadow, and three vertices of the feasible region are 

$$(x_1, x_2, x_3, x_4) = (0, \frac{5}{9}, \frac{1}{9}, 0), (x_1, x_2, x_3, x_4) = (0, 0, \frac{2}{9}, \frac{5}{9}), \text{ and } (x_1, x_2, x_3, x_4) = (\frac{5}{9}, 0, \frac{2}{9}, 0).$$

\begin{itemize}
  \item **Common mistake 1:** Not projecting the problem into the inequality form involving two variables.
  \item **Common mistake 2:** Not enumerating the coordinates of the extreme vertices of the feasible set.
\end{itemize}

2.2

[2 pts] On the same diagram, draw the objective function direction.

\begin{itemize}
  \item **Solution:** Look at the figure above.
\end{itemize}

\(^1\)This nice picture is made by Yuandong Tian.
Common mistake 3: If you did not project the problem into the inequality form involving two variables, you will have difficulty in drawing the direction of the objective function, since you need to project it onto the 2-dimensional space.

2.3

[3 pts] Solve the LP geometrically, by looking at the diagram. State the optimal value and the solution(s).

★ Solution: The bottom left corner of the feasible region is our optimal solution \((x_1, x_2, x_3, x_4) = (0, \frac{5}{4}, \frac{1}{4}, 0)\), and the optimal value of the objective is 1.5.

Common mistake 4: Use \(x_3\) instead of \(2x_3 + 1\) as the objective.

3 Rational Objective Function [Yi, 15 + 5 points]

We consider problems where the objective function is rational, and the numerator and the denominator are both linear functions of \(x\). The constraints are still linear. Some applications include:

- Maximizing the profit per unit time, rather than the total profit.
- Maximizing the smallest signal to noise ratio in a transmitter-receiver scenario.
- Obtaining confidence intervals i.e. upper and lower bounds while doing parameter estimation using frequentist techniques.

Consider the following problem:

\[
\begin{align*}
\min_x & \quad \frac{e^T x + d}{e^T x + f} \\
\text{subject to} & \quad Ax \leq b \\
& \quad e^T x + f > 0
\end{align*}
\]

3.1

[7 pts] Assume that the feasible region defined by the constraints is bounded, and the value of the optimal solution lies in the interval \([L, U]\). Further assume that for all points \(x\) in the feasible region, \(e^T x + f \geq \epsilon\), for some fixed \(\epsilon > 0\). Show that we can compute the optimal value of this problem to any desired accuracy using a series of linear programs.

★ Solution: An optimization problem where the constraints are linear and the objective function is a ratio of two linear functions is known as a linear fractional program.

Consider the problem of deciding whether the optimal value for the fraction \(\frac{e^T x + d}{e^T x + f}\) is less than or equal to a certain number \(K \in [L, U]\).

\[
\frac{e^T x + d}{e^T x + f} \leq K \\
\implies e^T x + d \leq K(e^T x + f) \quad \text{(since } e^T x + f > 0) \tag{5}
\]

We now consider the following feasibility problem

\[
\text{find } x
\]
subject to \[ Ax \leq b \]
\[ c^T x + d \leq K(e^T x + f) \]
\[ e^T x + f \geq \epsilon \] (6)

where the last equation follows from the assumption in the question. If the above LP is feasible, then it implies that there exists an \( x \) such that the point \( x \) is feasible with respect to the original constraints, and has a value of the fraction less than \( K \). This implies that the optimal value is between \([L,K]\). If on the other hand the above LP is infeasible, it implies that there exists no feasible point with a value for the fraction lower than \( K \). Since we know the optimal was between \([L,U]\), this implies that the optimal value lies in \([K,U]\).

To get an estimate of the optimal value to any desired degree of accuracy \( \delta \), we do binary search over the interval \([L,U]\) as follows:

1. Let \( K = \frac{L+U}{2} \).
2. Solve the feasibility problem 6.
3. If feasible, set \( U = K \), else set \( L = K \).
4. If \( \delta < \frac{U-L}{2} \), return the optimal value to be \( \frac{L+U}{2} \). Else, go to step 1.

\[ \text{Common mistake 1: You can not use the constraint } e^T x + f > 0 \text{ directly, since it is a strict inequality constraint. This is one of the points that this question is trying to make.} \]

\[ \text{Common mistake 2: The added constraint in each iteration of the binary search is } c^T x + d \leq K(e^T x + f), \text{ not } “=“ \text{ or } “\geq“ \].

3.2

[5 pts] Let us now remove the assumption that \( e^T x + f \geq \epsilon \), so that now we have a strict inequality \( e^T x + f > 0 \). How can we solve the problem now?

\[ \text{Solution: If we remove the assumption that } e^T x + f \geq \epsilon, \text{ then we are left with the constraint } e^T x + f > 0. \] Since an LP cannot handle a strict inequality, let us relax this constraint to \( e^T x + f \geq 0 \). The LP 6 now becomes

\[ \text{find } x \]
\[ \text{subject to } Ax \leq b \]
\[ c^T x + d \leq K(e^T x + f) \]
\[ e^T x + f \geq 0 \] (7)

However, now we need to deal with the case where the feasible solution makes the denominator \( e^T x + f = 0 \). It is possible that the LP above (7) returns a value of \( x \) such that the denominator is 0. Moreover, if there exists a point \( x' \) such that \( Ax' \leq b \) and \( c^T x' + d = e^T x' + f = 0 \), then this point will be feasible for any value of \( K \). Clearly this point was not part of the original feasible region (since in the original feasible region \( e^T x + f > 0 \)), but it will belong to the feasible region of LP 7. Hence, we cannot use the LP 7 to decide whether the fraction \( \frac{c^T x + d}{e^T x + f} \) is less than or equal to a certain number \( K \). To avoid this, use the following LP, which has a special objective:

\[ \max_x e^T x + f \]
subject to  
\[ Ax \leq b \]
\[ c^T x + d \leq K(e^T x + f) \]
\[ e^T x + f \geq 0 \]  \hspace{1cm} (8)

Suppose when we solve the above LP (8), we indeed get a solution \( x' \) such that \( e^T x' + f = 0 \). This means that there is no point \( x \) can satisfy all the constraints and \( e^T x + f > 0 \), since we are trying to maximize \( e^T x + f \). If this LP 8 returns a feasible value of \( x \) and for that value of \( x \), \( e^T x + f > 0 \), then we know the optimal value of \( \frac{e^T x + f}{e^T x + f} \) is less than or equal to \( K \) subject to the given constraints.

The last thing we need to ensure is that the above LP is bounded. It is given that the original problem has a bounded feasible region. For the feasible region of the current LP, since we are maximizing \( e^T x + f \) in the objective, adding \( e^T x + f = 0 \) can not render the LP unbounded. Therefore, the above LP is bounded.

Now, we have a procedure to test whether the fraction \( \frac{e^T x + f}{e^T x + f} \) is less than or equal to a certain number \( K \). We can now use binary search to get the value of the optimal solution to any desired accuracy.

■ Common mistake 1: Forget to mention that the proposed LP is bounded.

3.3

[3 pts] Provide a simple approach for obtaining some upper bound \( U \) on the optimal value, in polynomial time.

Hint: This should be easy.

★ Solution: Since this is a minimization problem, any feasible solution will provide a valid upper bound. We can find a feasible solution as follows

\[ \max \quad e^T x + f \]
subject to  
\[ Ax \leq b \]
\[ e^T x + f \geq 0 \]  \hspace{1cm} (9)

If the above optimization problem returns a value \( x' \), such that \( e^T x' + f = 0 \), then there is no feasible point for the original problem (but this will not happen if we assume the feasible region of the original problem is not empty). Also, this problem is bounded, since the original problem has a bounded feasible region and we only add \( e^T x + f = 0 \).

■ Common mistake 1: It is not sufficient to state that any feasible point provides a valid upper bound. You need to explicitly show how can we obtain a feasible point. Moreover, you can not use the constraint \( e^T x + f > 0 \), since we can not use a linear program to handle strict inequalities.

■ Common mistake 2: Forget to mention that the proposed LP is bounded.

3.4

[Extra credit, 5 pts] Provide a simple approach for obtaining some lower bound \( L \) in polynomial time.

Hint: This might be hard.
Solution: We first try to minimize the numerator:

$$\min_x c^T x + d$$

subject to

$$Ax \leq b$$
$$e^T x + f \geq 0$$

(10)

If this problem is not feasible, then the original fractional programming is infeasible, either. So let's assume this is feasible. Also, this problem is bounded, since the original feasible region is bounded and we here use the closure of it.

Call the optimal objective as $C_{\min}$. If we achieve $C_{\min} \geq 0$, we conclude that the original fractional programming problem cannot achieve negative objective, so 0 is a safe lower bound (or in the case of $C_{\min} > 0$, you can also try to maximize the denominator as (9), denoted by $E_{\max}$, and get a tighter lower bound by $\frac{C_{\min}}{E_{\max}}$).

If we achieve an optimal value $C_{\min} < 0$, we go ahead to solve:

$$\min_x e^T x + f$$

subject to

$$Ax \leq b$$
$$e^T x + f \geq 0$$
$$c^T x + d \leq 0$$

We use $E_{\min}$ to denote the optimal objective. If $E_{\min} > 0$, we conclude that $\frac{C_{\min}}{E_{\min}}$ is a lower bound (note that $C_{\min} < 0$ and $E_{\min} > 0$). Otherwise, if we find $E_{\min} = 0$, we have an annoying case that $e^T x + f$ in the original problem may be able to approach zero infinitely close while keeping $c^T x + d$ negative, which makes the objective arbitrarily negative. In this case, we can use the LP (8) to conduct a search for exponentially decreasing $K$, e.g., $K = -1, -2, -4, -8, \ldots$, until we find an infeasible $K$, or we conclude that the objective of the original problem can be infinitely negative.

Common mistake 1: There is a smart way to solve the whole linear fractional programming question, as shown in the BV's textbook page 151. It is good to use that technique to solve our question, as long as you make your own connection between the textbook and each of our sub-questions (some students get full credits by doing so). But just copying the textbook will not get full credits for this question.

4 Linear Programming with Uncertainty [Yi, 15 points]

4.1 Hint 0

This isn't really a hint, but first show that $a_i \in \{\hat{a}_i + v \mid \|v\|_\infty \leq \epsilon\}$ is equivalent to $\|a_i - \hat{a}_i\|_\infty \leq \epsilon$.

Solution: Both directions of the equivalence can be proved using $v = a_i - \hat{a}_i$.

4.2 Hint 1

For $x, v \in \mathbb{R}^n$, show that $x^T v \leq \|x\|_1$ for all $v$ with $\|v\|_\infty \leq 1$. Also, is this inequality tight? i.e., is there $v$ that can satisfy $x^T v = \|x\|_1$?

Solution: Prove the inequality by

$$x^T v = \sum_j x_j v_j \leq \sum_i |x_j| |v_j| \leq \max_j |v_j| \sum_j |x_j| \leq \|v\|_\infty \sum_j |x_j| \leq \sum_j |x_j| = \|x\|_1.$$  

Also, this inequality is tight. For any $x$, let $v_j = \text{sign}(x_j)$, $j = 1, 2, \ldots, n$ and we will have $x^T v = \|x\|_1$. 

7
4.3 Hint 2

Using the result of Hint 1, show that the constraint \( \hat{a}_i^T x \leq b_i \) for all \( a_i \in \{ \hat{a}_i + v \mid \| v \|_\infty \leq \epsilon \} \) is actually equivalent to the following constraint: \( \hat{a}_i^T x + \epsilon \| x \|_1 \leq b_i \).

★ Solution: We first rephrase hint 1 in a scaled version: \( x^T v \leq \epsilon \| x \|_1 \) for all \( v \) with \( \| v \|_\infty \leq \epsilon \). And this is tight using a similar construction \( v_j = \epsilon \text{sign}(x_j), \ j = 1, 2, \ldots, n \).

Now, both directions of hint 2 need to be proved.

On the one hand, if \( a_i^T x \leq b_i \) for all \( a_i \in \{ \hat{a}_i + v \mid \| v \|_\infty \leq \epsilon \} \), we use a specific \( v^* \) such that the inequality in scaled version of hint 1 is tight, i.e., \( v_j^* = \epsilon \text{sign}(x_j), \ j = 1, 2, \ldots, n \). Then we will have \( \hat{a}_i^T x + \epsilon \| x \|_1 = \hat{a}_i^T x + x^T v^* = (\hat{a}_i + v^*)^T x \leq b_i \).

On the other hand, if \( \hat{a}_i^T x + \epsilon \| x \|_1 \leq b_i \), we directly use scaled version of hint 1: \( a_i^T x = (\hat{a}_i + v)^T x = \hat{a}_i^T x + x^T v \leq \hat{a}_i^T x + \epsilon \| x \|_1 \leq b_i \) for any \( \| v \|_\infty \leq \epsilon \).

4.4 Hint 3

Using the result of Hint 2, we can write the optimization problem (??) as minimizing a linear objective function with nonlinear constraints (since constraints involve the \( \ell^1 \) norm \( \| x \|_1 \)). Then, we can further convert the nonlinear program problem into an LP.

Using hint 2, we have

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \hat{a}_i^T x + \epsilon \| x \|_1 \leq b_i, \quad i = 1, 2, \ldots, m
\end{align*}
\]

This is a nonlinear program since we have \( \| x \|_1 = \sum_j |x_j| \) in the constraints. Now we can use \( n \) new variables \( \xi_j, \ j = 1, 2, \ldots, n \) to define an equivalent LP:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \hat{a}_i^T x + \epsilon \sum_{j=1}^n \xi_j \leq b_i, \quad i = 1, 2, \ldots, m \\
& \quad x_j \leq \xi_j, \quad j = 1, 2, \ldots, n \\
& \quad x_j \geq -\xi_j, \quad j = 1, 2, \ldots, n
\end{align*}
\]