Linear Programming: Formulations, Geometry and Simplex Method

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Outline

- Different forms of LPs
- Geometry of LPs
- Solving an LP: Simplex Method
- Summary
Inequality form of LPs

- An LP in inequality form \((x \in \mathbb{R}^n)\)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

- Matrix notation

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]
Why is inequality form useful?

- Intuitive: sketching an LP
- Understand the geometry of LPs

minimize $c^T x$
subject to $a_i^T x \leq b_i$, $i = 1, \ldots, m$
Standard form of LPs

- An LP in standard form
  \[
  \begin{align*}
  \text{minimize} & \quad c^T x \\
  \text{subject to} & \quad g_i^T x = h_i, \quad i = 1, \ldots, m \\
  & \quad x \geq 0
  \end{align*}
  \]

- Matrix notation
  \[
  \begin{align*}
  \text{minimize} & \quad c^T x \\
  \text{subject to} & \quad Gx = h \\
  & \quad x \geq 0
  \end{align*}
  \]
Why is standard form useful?

- Easy for computers to operate 😊
  - Search “corners” of the feasible region
  - Transform of constraints
  - E.g., simplex method works in standard form
Inequality form $\rightarrow$ standard form

- Add slack variables

\[
\begin{align*}
\text{min} & \quad 3x + 4y \\
\text{subject to} & \quad x + y \leq 5 \\
& \quad 3x + 5y \leq 13 \\
& \quad x, y \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad 3x + 4y \\
\text{subject to} & \quad x + y + s_1 = 5 \\
& \quad 3x + 5y + s_2 = 13 \\
& \quad x, y, s_1, s_2 \geq 0
\end{align*}
\]
Stanford form $\rightarrow$ inequality form

- Make and drop slack variables

\[
\begin{align*}
\min_{x,y,s_1,s_2} & \quad 3x + 4y \\
\text{subject to} & \quad 4x + 6y + s_1 + s_2 = 18 \\
& \quad 3x + 5y + s_2 = 13 \\
& \quad x, y, s_1, s_2 \geq 0
\end{align*}
\rightarrow
\begin{align*}
\min_{x,y,s_1,s_2} & \quad 3x + 4y \\
\text{subject to} & \quad x + y + s_1 = 5 \\
& \quad 3x + 5y + s_2 = 13 \\
& \quad x, y, s_1, s_2 \geq 0
\end{align*}
\rightarrow
\begin{align*}
\min_{x,y,s_1,s_2} & \quad 3x + 4y \\
\text{subject to} & \quad x + y = 5 \\
& \quad 3x + 5y + s_2 = 13 \\
& \quad x, y, s_1, s_2 \geq 0
\end{align*}
\rightarrow
\begin{align*}
\min_{x,y} & \quad 3x + 4y \\
\text{subject to} & \quad x + y \leq 5 \\
& \quad 3x + 5y \leq 13 \\
& \quad x, y \geq 0
\end{align*}
General form of LPs

- An LP in general form

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m \\
& \quad g_i^T x = h_i, \quad i = 1, \ldots, p
\end{align*}
\]

- Transform to
  - Inequality form: sketching, geometry
  - Standard form: simplex method
Outline

- Different forms of LPs
- **Geometry of LPs**
  - Half space and polyhedron
  - Extreme points, vertices and basic feasible solution
  - Optimality of LPs at extreme points
- Solving an LP: Simplex Method
- Summary
Half space and polyhedron

- An inequality constraint $\rightarrow$ a *half space*
- A set of inequality constraints $\rightarrow$ a *polyhedron*

$$\mathcal{P} = \{x \mid Ax \leq b\} = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$$

[Boyd & Vandenberghe]
Geometry of LPs

An LP in inequality form \((x \in \mathbb{R}^n)\)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

[Boyd & Vandenberghe]
Geometry of LPs

- An LP in inequality form \((x\ in \mathbb{R}^n)\)
  
  \[
  \begin{align*}
  &\text{minimize} & \quad c^T x \\
  &\text{subject to} & \quad a_i^T x \leq b_i, & \quad i = 1, \ldots, m
  \end{align*}
  \]

- Also, an LP can be
  - Infeasible
  - Unbounded
Geometry of LPs

- Three important concepts of an LP
  - Extreme points
  - Vertices
  - Basic feasible solutions

[Boyd & Vandenberghe]
Concept 1: extreme points

- A point $x$ in $P$ is an **extreme point**:
  - It cannot be represented as
    \[ x = \lambda y + (1 - \lambda)z, \quad 0 \leq \lambda \leq 1, \ y, z \in P, \ y \neq x, \ z \neq x \]
  - Not in the middle of any other two points in $P$

[Boyd & Vandenberghe]
Concept 2: vertices

- A point $x$ in $P$ is a **vertex**:
  - It is *uniquely* optimal for some objective function

[Boyd & Vandenberghe]
Concept 3: Basic feasible solutions

- An inequality constraint is **active** at $x$:
  - The constraint holds with equality at $x$

$$\mathcal{P} = \{x \mid Ax \leq b\} = \{x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m\}$$

[Boyd & Vandenberghe]
Concept 3: Basic feasible solutions

- A point $x$ is a **basic solution**:
  - There exist $n$ linearly independent active constraints at $x$

\[ \mathcal{P} - \{ x \mid Ax \leq b \} - \{ x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m \} \]

[Boyd & Vandenberghe]
Concept 3: Basic feasible solutions

- A point $x$ is a **basic feasible solution**: A basic solution that satisfies all constraints (i.e., stay in $P$)

$$
P = \{ x \mid Ax \leq b \} - \{ x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m \}$$

[Boyd & Vandenberghe]
Equivalence of three definitions

- Extreme points, vertices and basic feasible solution are *equivalent*
  - Extreme points: not in the middle of any two
  - Vertices: uniquely optimal for some objective
  - Basic feasible solutions: n indep. active constraints

- Intuition of proofs
  - Vertex $\rightarrow$ extreme point
  - Extreme point $\rightarrow$ basic feasible solution
  - Basic feasible solution $\rightarrow$ vertex
Why are these definitions useful?

- Equivalent ways to define “corners”
  - Extreme points
  - Vertices
  - Basic feasible solutions
- Optimality of LPs at “corners”
Optimality of extreme points

- Given an LP

- If
  - The polyhedron $P$ has at least one extreme point
  - Optimal solutions exist (not unbounded or infeasible)

- Then
  - At least one optimal solution is an extreme point
Search basic feasible solutions!

- Solve an LP: search over extreme points
- Extreme points ↔ basic feasible solutions
  - Search over basic feasible solutions!
  - Basic idea of simplex method
Outline

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Search basic feasible solutions

- Optimality of extreme points
- Extreme points $\leftrightarrow$ basic feasible solutions
- Solve LP: search over basic feasible solutions!
Search basic solutions in standard form

- Simplex method operates in *standard* form
  - Understand the geometry in *inequality* form 😊
  - Search basic solutions in *standard* form ?
Inequality form vs. standard form

- \( \text{max } 2x + 3y \text{ s.t.} \)
  - \( x + y \leq 4 \)
  - \( 2x + 5y \leq 12 \)
  - \( x + 2y \leq 5 \)
  - \( x, y \geq 0 \)

- or s.t.
  - \( x + y + u = 4 \)
  - \( 2x + 5y + v = 12 \)
  - \( x + 2y + w = 5 \)
  - \( x, y, u, v, w \geq 0 \)
Search basic solutions in standard form

Consider the standard-form LP problem, \( x \in \mathbb{R}^n \):

\[
\begin{align*}
\text{min } & c^T x \\
\text{s.t. } & Ax = b \\
& x \geq 0
\end{align*}
\]

where \( A \) is \( m \times n \) matrix, with \( n \geq m \).

- How to get a basic solution in standard form?
  - Pick a basis (\( m \) independent columns)
  - Fix the rest (\( n-m \)) non-basic vars to 0
  - Solve for \( m \) basic vars
Search basic solutions in standard form

Example

\[ \begin{align*}
A &= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 5 & 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \end{bmatrix}
\end{align*} \]

\[
\begin{align*}
1 & \quad 1 & \quad 0 & \quad 0 & \quad 4 \\
2 & \quad 5 & \quad 0 & \quad 1 & \quad 0 & \quad 12 \\
1 & \quad 2 & \quad 0 & \quad 0 & \quad 1 & \quad 5 \\
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} &= 4 \\
\begin{bmatrix} 2 & 5 & 0 & 1 & 0 \end{bmatrix} &= 12 \\
\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \end{bmatrix} &= 5
\end{align*}
\]

\[
\begin{align*}
\text{set } x, y = 0 & \quad u = 4, \ v = 12, \ w = 5 \\
\text{feasible } & \quad \text{basic solution}
\end{align*}
\]

\[
\begin{align*}
\text{set } v, w = 0 & \quad \{x, y, u\} = 1, 2, 1 \\
\end{align*}
\]

\[
\begin{align*}
\{0, 0, u, 12, 5\} & \quad \text{basic solution} \\
\end{align*}
\]

\[
\begin{align*}
\{x, y, u\} = 1, 2, 1 \\
\begin{bmatrix} 1 \ 2 \ 1 \ 0 \ 0 \end{bmatrix}
\end{align*}
\]

Diagram showing the solution space and feasible region.
Trick: monitor the objective function during the search

- add another variable $z$ (unbounded) to represent objective
- add a constraint $z - c^T x = 0$
- keep $z$ in basis at all times
- $A(:,[1 4 5 6]) \setminus [A b]$

max $z = 2x + 3y$ s.t.

\[
\begin{align*}
\begin{array}{cccccccc}
\text{RHS} & x & y & u & v & w & z & \\
4 & 1 & 1 & 1 & 0 & 0 & 0 & \\
12 & 2 & 5 & 0 & 1 & 0 & 0 & \\
5 & 1 & 2 & 0 & 0 & 1 & 0 & \\
4 & -2 & 1 & 0 & 0 & 0 & 1 & \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & \\
8 & 2 & 0 & 0 & 1 & 0 & 0 & \\
\end{array}
\end{align*}
\]
Simplex method

- Simplex method
  - Search over basic feasible solutions
  - Repeatedly move to a *neighbor* bfs to *improve* objective
  - Stop at “local” optimum
Simplex method: an example

- Maximize \( Z = 5x_1 + 2x_2 + x_3 \)
  
  \[
  x_1 + 3x_2 - x_3 \leq 6, \\
  x_2 + x_3 \leq 4, \\
  3x_1 + x_2 \leq 7, \\
  x_1, x_2, x_3 \geq 0.
  \]
Simplex method: an example

- Maximize $Z = 5x_1 + 2x_2 + x_3$
  \[ x_1 + 3x_2 - x_3 + x_4 = 6, \]
  \[ x_2 + x_3 + x_5 = 4, \]
  \[ 3x_1 + x_2 + x_6 = 7, \]
  \[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \]

- Go through the example …
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- Summary
Summary

- Different forms of LPs
  - Inequality, standard, general ..
- Geometry of LPs
  - Focus on Inequality form LPs
  - Half space and polyhedron
  - Extreme points, vertices and basic feasible solutions – three definitions of “corners”
  - Optimality at “corners”
Summary

- Simplex method
  - Operate in standard form
  - Search over “corners”
  - Start from a basic feasible solution (i.e., a basis)
  - Search over neighboring basis
    - Improve the objective
    - Keep feasibility
  - Stop at local(?) optimum