Convex Functions, Convex Sets and Quadratic Programs

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Outline

- Convex sets
  - Definitions
  - Motivation
  - Operations that preserve set convexity
  - Examples

- Convex Function
  - Definition
  - Derivative tests
  - Operations that preserve convexity
  - Examples

- Quadratic Programs
Quick definitions

- Convex set
  - For all $x,y$ in $C$: $\theta x + (1- \theta) y$ is in $C$ for $\theta \in [0,1]$

- Affine set
  - For all $x,y$ in $C$: $\theta x + (1- \theta)y$ is in $C$
  - All affine sets are also convex

- Cones
  - For all $x$ in $C$: $\theta x$ is in $C$ $\theta \geq 0$
  - Convex cones: For all $x$ and $y$ in $C$, $\theta_1 x + \theta_2 y$ is in $C$
Why do we care about convex and affine sets?

- The basic structure of any convex optimization
  - \( \min f(x) \) where \( x \) is in some convex set \( S \)

- This might be more familiar
  - \( \min f(x) \) where \( g_i(x) \leq 0 \) and \( h_i(x) = 0 \)
  - \( g_i \) is convex function and \( h_i \) is affine

- Cones relate to something called Semi Definite Programming which are an important class of problems
Operations that preserve convexity of sets

- Basic proof strategy
- Ones we saw in class – let's prove them now
  - Intersection
  - Affine
  - Linear fractional
- Others include
  - Projections onto some of the coordinates
  - Sums, scaling
  - Linear perspective
Quick review of examples of convex sets we saw in class

- Several linear examples (halfspaces (not affine), lines, points, $\mathbb{R}^n$)
- Euclidean ball, ellipsoid
- Norm balls (what about $p < 1$?)
- Norm cone – are these actually cones?
Some simple new examples

- Linear subspace – convex
- Symmetric matrices - affine
- Positive semidefinite matrices – convex cone
- Let's go over the proofs!!
Convex hull

- Definition

- Important lower bound property in practice for non-convex problems – the two cases

- You’ll see a very interesting other way of finding “optimal” lower bounds (duality)
Convex Functions

- Definition
  - $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$

- Alternate definition in terms of epigraph
  - Relation to convex sets
Proving a function is convex

- It’s often easier than proving sets are convex because there are more tools
  - First order
    - Taylor expansion (always underestimates)
    - Local information gives you global information
    - Single most beautiful thing about convex functions
  - Second order condition
    - Quadratics
    - Least squares?
Some examples without proofs

- In $\mathbb{R}$
  - Affine (both convex and concave function) unique
  - Log (concave)

- In $\mathbb{R}^n$ and $\mathbb{R}^{mxn}$
  - Norms
  - Trace (generalizes affine)
  - Maximum eigenvalue of a matrix

- Many many more examples in the book
  - log sum exp, powers, fractions
Operations that preserve convexity

- Nonnegative multiples, sums
- Affine Composition $f(Ax + b)$
- Pointwise sup – equivalent to intersecting epigraphs
  - Example: $\text{sum}(\max_{1 \ldots r}[x])$
  - Pointwise inf of concave functions is concave
- Composition
- Some more in the book
Quadratic Programs

- Basic structure
- What is different about QPs?
- Lasso QP