• QP examples: Huber regression, LASSO

\[
\min_{a,b} (z + a - b)^2 + 2a + 2b \\
\text{s.t. } a, b \geq 0
\]

\[
\min_{w,s} \sum (y_i - x_i'w)^2 + \lambda \sum s_j \\
\text{s.t. } s_j \geq w_j, s_j \geq -w_j
\]
Support vector machines
(separable case)

\[ \min_{\mathbf{v}, d} \frac{1}{2} \|\mathbf{v}\|^2 \quad \text{s.t.} \quad y_i (\mathbf{x}_i \cdot \mathbf{v} - d) \geq 1 \]
Review: duality

• Note = constraint
  \[\begin{align*}
  \min \ & x - 2y \\
  \text{s.t.} \ & a(x + y \geq 2) \\
  & x, y \geq 0 \\
  & d(3x + y = 2)
  \end{align*}\]

• Use multipliers to write combined constraints

• Constrain multipliers to give us a bound on objective

• Optimize to get tightest bound
Review: duality

• Primal feasible $\geq$ primal opt:
  ▷

• Dual opt $\geq$ dual feasible:
  ▷

• Primal feasible ($\&$ opt) $\geq$ dual feasible ($\&$ opt):
  ▷

• If primal opt = dual opt:
  ▷
Matrix form

• Primal:
  ‣ \( \min_x c'x \) s.t. \( Ax \geq b \)

• Multipliers:

• Linear comb. of constrs:

• Lower bound on obj:

• Dual:
Dual dual

• Dual:
  ‣ $\max_y b'y$ s.t. $A'y = c, y \geq 0$

• Multipliers:

• Linear comb. of constrs:

• Upper bound on obj:

• Primal:
Geometric interpretation

- min 3y s.t.
  - \(-x + 2y \geq 0\)
  - \(x + y \geq 2\)
Geometric interpretation

- \( \min x + 3y \) s.t.
  - \(-x + 2y \geq 0\)
  - \(x + y \geq 2\)
Geometric interpretation

- $a \cdot \text{constr}_1 + b \cdot \text{constr}_2$

- Do this until:

- In dimension $d$
More constraints

• min \( x + 3y \) s.t.
  - \(-x + 2y \geq 0\)
  - \(x + y \geq 2\)
  - \(-2x + y \geq -4\)

• Interpret: 0 dual variable =
Complementary slackness

• Primal-dual pair:
  ‣ \( \min c'x \text{ s.t. } Ax \geq b \)
  ‣ \( \max b'y \text{ s.t. } A'y = c, y \geq 0 \)

• Let \( x \) be primal optimal, \( y \) be dual optimal
  ‣ define \( s = \)

• Then: \textbf{at most one} of or

• Usual statement:
LP duality cheat sheet

Min $c'x + d'y$ s.t. $Ax + By \geq p$
Ex + Fy = q
x free, $y \geq 0$

Max $p'v + q'w$ s.t. $A'v + E'w = c$
$B'v + F'w \leq d$
v $\geq 0$, w free

Swap RHS and objective
Swap max/min

Transpose constraint matrix
+ve vars yield $\leq$, free vars yield =
## LP duality summary

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>min problem</td>
<td></td>
</tr>
<tr>
<td>max problem</td>
<td></td>
</tr>
<tr>
<td>constraint</td>
<td></td>
</tr>
<tr>
<td>( \leq ) constraint</td>
<td></td>
</tr>
<tr>
<td>( \geq ) constraint</td>
<td></td>
</tr>
<tr>
<td>( = ) constraint</td>
<td></td>
</tr>
<tr>
<td>variable</td>
<td></td>
</tr>
</tbody>
</table>
**LP duality summary**

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>tight constraint</td>
<td></td>
</tr>
<tr>
<td>slack constraint</td>
<td></td>
</tr>
<tr>
<td>zero/nonzero variable</td>
<td></td>
</tr>
<tr>
<td>infeasible problem</td>
<td></td>
</tr>
<tr>
<td>unbounded problem</td>
<td></td>
</tr>
<tr>
<td>finite optimal value</td>
<td></td>
</tr>
</tbody>
</table>
What about QP duality?

• min $x^2 + y^2$ s.t.
  
  $x + 2y \geq 2 \quad x, y \geq 0$

• How can we lower-bound OPT?
  
  ‣ recall: supporting hyperplane thm: for $x$ on boundary of convex set $C$,
  
  ‣ apply to epigraph: $\{ (x, z) | z \geq f(x) \}$
What about QP duality?

• \( \min \, x^2 + y^2 \) \ s.t. \[ x + 2y \geq 2 \quad x, y \geq 0 \]

• How can we lower-bound OPT?
Can bound at any point

• \( \min x^2 + y^2 \) s.t.
  
  \[ x + 2y \geq 2 \quad x, y \geq 0 \]

• Try Taylor @ \((x, y) = (v, w)\)
SVM duality

- Recall: $\min \quad \text{s.t.}$
- Taylor bound objective:
- Linear comb. of constraints:

- To get bound, need:

- So, $\|w\|^2/2 \geq$
- Dual: $\max \quad \text{s.t.}$
Interpreting SVM dual

- \( \max_{\alpha, v} \sum_i \alpha_i - \|v\|^2/2 \) s.t.
  - \( \sum_i \alpha_i y_i = 0 \)
  - \( \sum_i \alpha_i y_i x_{ij} = v_j \quad \forall j \)
  - \( \alpha_i \geq 0 \quad \forall i \)

What are optimal \( \alpha, v \)?
Optimal b
Example: SVMs
Convex indicator functions

• $C$ a convex set

• $l_C(x) = \begin{cases} 
\text{convex: } & l_C(\lambda x + (1-\lambda)y) \leq \lambda l_C(x) + (1-\lambda)l_C(y) \\
\text{x, y in C: } & \\
\text{x, y not in C: } & \\
\text{x in C, y not in C: } & 
\end{cases}$
Ex: indicators of subspaces, polyhedra

• $C = \{ x \mid Ax = b \}$  \hspace{1cm} $D = \{ x \mid Ax \geq b \}$

• $f(x) = \max_y y'(Ax-b)$
  ‣ suppose $Ax = b$:  
  ‣ suppose $(Ax)_i \neq b_i$:  

• $g(x) = \max_{y \geq 0} y'(b-Ax)$
  ‣ suppose $Ax \geq b$:  
  ‣ suppose $(Ax)_i < b_i$:  

The Lagrangian

- $L(a, b, c, x, y) = [3x + y] - [a(x + y - 2) + bx + cy]$

- $\max_{a, b, c \geq 0} \min_{x, y} L(a, b, c, x, y)$
  - define: $M(a, b, c) =$

\[
\begin{align*}
\min & \quad 3x + y \quad \text{s.t.} \\
& \quad x + y \geq 2 \\
& \quad x, y \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \quad 2a \quad \text{s.t.} \\
& \quad a + b = 3 \\
& \quad a + c = 1 \\
& \quad a, b, c \geq 0
\end{align*}
\]
The Lagrangian

- \( L(a,b,c,x,y) = \) 
  
  \[ [3x + y] - [a (x + y - 2) + bx + cy] \]

- \( \min_{x,y} \max_{a,b,c \geq 0} L(a,b,c,x,y) \)
  
  - define: \( N(x,y) = \)
Saddle-point picture

- min \( y \) s.t. \( y \geq 2 \)
+ vs – in Lagrangian

\[ \begin{align*}
\min y \quad \text{s.t.} \quad 2 \leq y \leq 4 \\
\max y \quad \text{s.t.} \quad 2 \leq y \leq 4
\end{align*} \]
Example: max flow

- Given a directed graph
  - edges \((i,j) \in E\)
  - flows \(f_{ij}\), capacities \(c_{ij}\)
  - source \(s\), terminal \(t\) (\(c_{ts} = \infty\))
- \(\text{max } f_{ts}\)
Example: max flow

- max $f_{ts}$ s.t.
  - positive flow
  - capacity
  - flow conservation

\[
\begin{array}{ccccccc}
sx & sy & xy & xt & yt & ts \\
-1 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 \\
\end{array}
\]
Dual of max flow
Interpreting dual
min cut: image segmentation
Solution
The graduate student nutrition problem

\[
\begin{align*}
\text{min} & \quad 3x + y \quad \text{s.t.} \\
& x + y \geq 2 \\
& x \geq 0 \\
& y \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 2a \quad \text{s.t.} \\
& a + b = 3 \\
& a + c = 1 \\
& a, b, c \geq 0
\end{align*}
\]
Sensitivity analysis

\[
\begin{align*}
\text{min} & \quad 3x + y \quad \text{s.t.} \\
& x + y \geq 2 \\
& x \geq 0 \\
& y \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 2a \quad \text{s.t.} \\
& a + b = 3 \\
& a + c = 1 \\
& a, b, c \geq 0
\end{align*}
\]