LP vs. linear feasibility

- \( \min c^T x \) s.t. \( Ax \geq b \)
  - dual: \( \max b^T y \) s.t. \( A^T y = c, y \geq 0 \)
- find \( x \) s.t. \( Ax \geq b \)

find \( x, y \) s.t.
\[
\begin{align*}
Ax & \geq b \\
A^T y & = c, \quad y \geq 0 \\
b^T y & = c^T x
\end{align*}
\]
Reminder: separation oracle
Simplified preview: ellipsoid

find $x$ s.t. $Ax \geq b$
Difficulties

• How do we get bounding sphere? [later]

• How do we know when to stop?
  ‣ modify problem: $A x + 6 + \eta^T z \geq 0$
  ‣ $\eta$: too small to make feasible, large crash to get min volume
  ‣ stop when: go below min volume $\Rightarrow$ declare infeasible

• how do we get $\eta$? [later]

• Bound region gets complicated—how do we find its center?
Simplifying bounding region
Bounding a half-sphere

- We’ll do simple case 1st, solve harder cases by reducing to simple one
  - $E = \{ x \mid \| x \| \leq 1 \}$
  - $H = \{ x \mid x_1 \geq 0 \}$
- Ellipse: 
  \[
  \frac{(x-c)^2}{\rho^2} + \frac{y^2}{\sigma^2} \leq 1
  \]
  \[
  \frac{(1-c)^2}{\rho^2} = 1 \quad \rho^2 = (1-c)^2
  \]
  \[
  \frac{c^2}{\rho^2} + \frac{1}{\sigma^2} = 1 \quad \frac{1}{\sigma^2} = 1 - \frac{c^2}{(1-c)^2}
  \]
  \[
  = \frac{1 - 2c}{(1-c)^2}
  \]
Minimum area ellipse

- $\rho^2 = (1-c)^2$  $\sigma^2 = (1-c)^2/(1-2c)$
- Area $\propto \rho \sigma$
- $\min \ln(\text{area}^2) = \text{const} + \frac{c}{1-c} - \ln(1-2c)$
- $\frac{d}{dc} \left( \frac{c}{1-c} \right) = -\ln(1-c) + 2\sqrt{1-2c}
- \frac{d}{dc} \left( \ln(1-2c) \right) = 2 \frac{1}{1-2c}
- 2 \leq c \leq 1/3$
In n dimensions

- Still: $\rho^2 = (1-c)^2 \quad \sigma^2 = (1-c)^2/(1-2c)$
- Volume $\propto \rho \sigma^{n-1}$
- \( \min \ln(\text{volume}^2) = \text{const} + 2 \ln(1-c) + (n-1)2 \ln(1-c) - (n-1) \ln(1-2c) \)

\[ c = \frac{1}{n+1} \]
Bounding a half-ellipsoid

- General ellipsoid w/ center $x_C$, shape $A$:
  \[ E = \{ x \mid (x-x_c)^T A (x-x_c) \leq 1 \} \]
  \[ A = u^T u \]

- Halfspace: $p^T x \leq p^T x_C$

- Translate to origin, scale to be spherical
  \[ y = u^T (x-x_c) \quad x = u^{-1} y + x_c \]

- Rotate so hyperplane is axis-normal
Ellipsoid algorithm

to find $x$ s.t. $Ax + b + \eta \geq 0$

• Pick $E_0$ s.t. $x^* \in E_0$

• for $t := 1, 2, \ldots$
  ‣ $x_t :=$ center of $E_t$
  ‣ ask oracle whether $Ax_t + b + \eta \geq 0$
  ‣ yes: declare feasible!
  ‣ no: get new constraint w/ normal $p_t$
  ‣ $E_{t+1} := \text{bound}(E_t \cap \{ x \mid p_t^T x \leq p_t^T x_t \})$
  ‣ if $\text{vol}(E_{t+1}) \leq \varepsilon \text{vol}(E_0)$: declare infeasible!
Ellipsoid, graphically
How many iterations?

\[ \rho = \frac{n}{n+1} \quad \sigma = \sqrt{\frac{n^2}{n^2 - 1}} \]  

\[ \ln(1+x) \leq x \]

- new vol \( \propto \rho \sigma^{n-1} \)
- \( \ln(\text{new vol} / \text{orig vol}) = \ln(\rho \sigma^{n-1}) \)
  
\[ \ln\left(1 - \frac{1}{n+1}\right) + \frac{(n-1)}{2} \ln\left(1 + \frac{1}{n^2-1}\right) \]

\[ \leq -\frac{1}{n+1} + \frac{n-1}{2} \cdot \frac{1}{n^2-1} = -\frac{1}{2} \cdot \frac{1}{n+1} \]

\[ e^{-\frac{T}{2} \frac{1}{n+1}} \leq e^{\frac{T}{n+1}} \]

\( T \geq 2(n+1) \frac{1}{n^2} \)
How big, how small?

• Need $E_0$ guaranteed to contain a feasible solution (if one exists), and $\eta$ small enough not to affect feasibility

• Suppose all coefficients are M-bit integers

• Then basic solutions: can’t be too big or too close together
How big, how small?

\[ M = 4 \]

\[ y \leq 1 \]

\[ y - y \geq x \]

\[ 8x + 8y \geq 7 \]

\[ 8x + 8y \geq 8 \]
How small: zoomed
Including $\eta$
How big and how small?

• \(Ax + b \geq 0\) (n vars, m constrs, bitlength \(M\))

• Basic solution: \(C_x + d = 0\) \(C = A(\text{bas}, i)\) \(d = b(\text{bas})\)

• Cramer’s rule: \(x_i = -\frac{|C_i|}{|C|}\)

• Bit length of \(|C| \leq R = O(\ln n + Mn)\)

• So: radius of \(E_0 = 2^R \sqrt{n}\) volume \(\leq (2 \cdot 2^R \sqrt{n})^n\)

• \(\eta = \frac{1}{2} 2^{-R}\) min vol \(\geq 2^{-R \ln n}\)

• \(\log(\varepsilon) \leq n (\log 2 + R \log 2 + \frac{1}{2} \log n) + n (R+1) \log 2\)

\(\frac{1}{2} \log n\)
Gotchas

• What if original LF problem was unbounded?
  so what?

• Numerical precision
  $O \left( n^3 M \right) \text{ bits}$