LP vs. linear feasibility

• min $c^T x$ s.t. $Ax \geq b$
  - dual: max $b^T y$ s.t. $A^T y = c, y \geq 0$

• find $x$ s.t. $Ax \geq b$
Reminder: separation
oracle
Preview: ellipsoid

find $x$ s.t. $Ax \geq b$
Difficulties

• How do we get bounding sphere?
• How do we know when to stop?
  ‣ modify problem:
  ‣ $\eta$:
  ‣ stop when:
  ‣ how do we get $\eta$?
• Bound region gets complicated—how do we find its center?
Simplifying bounding region
Bounding a half-sphere

• We’ll do simple case 1st, solve harder cases by reducing to simple one
  ▸ E =
  ▸ H =

• Ellipse:
Minimum area ellipse

- $\rho^2 = (1-c)^2$  $\sigma^2 = (1-c)^2/(1-2c)$
- Area
- $\min \ln(\text{area}^2) =$
In n dimensions

- Still: $\rho^2 = (1-c)^2 \quad \sigma^2 = (1-c)^2/(1-2c)$
- Volume
- $\min \ln(\text{volume}^2) =$
Bounding a half-ellipsoid

- General ellipsoid w/ center $x_C$, shape $A$:

- Halfspace: $p^T x \leq p^T x_C$

- Translate to origin, scale to be spherical

  $y = \quad x = \quad$

- Rotate so hyperplane is axis-normal
Ellipsoid algorithm

to find $x$ s.t. $Ax + b + \eta \geq 0$

- Pick $E_0$ s.t. $x^* \in E_0$

- for $t := 1, 2, \ldots$
  - $x_t :=$ center of $E_t$
  - ask oracle whether $Ax_t + b + \eta \geq 0$
  - yes: declare feasible!
  - no: get new constraint w/ normal $p_t$
  - $E_{t+1} :=$ bound($E_t \cap \{ x \mid p_t^T x \leq p_t^T x_t \}$)
  - if $\text{vol}(E_{t+1}) \leq \varepsilon \text{vol}(E_0)$: declare infeasible!
Ellipsoid, graphically
How many iterations?

\[ \rho = \frac{n}{n + 1} \quad \sigma = \sqrt{\frac{n^2}{n^2 - 1}} \quad \ln(1 + x) \leq x \]

- new vol
- \( \ln(\text{new vol} / \text{orig vol}) = \)
How big, how small?

• Need $E_0$ guaranteed to contain a feasible solution (if one exists), and $\eta$ small enough not to affect feasibility

• Suppose all coefficients are M-bit integers

• Then basic solutions:
How big, how small?
How small: zoomed
Including $\eta$
How big and how small?

• $Ax + b \geq 0$  (n vars, m constrs, bitlength M)

• Basic solution:

• Cramer’s rule:

• Bit length of $|C| \leq$

• So: radius of $E_0 = \text{volume} \leq$

• $\eta = \text{min vol} \geq$

• $\log(\varepsilon) \leq$
Gotchas

• What if original LF problem was unbounded?

• Numerical precision