The geometry of LP solutions

Optimization - 10725
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The Simplex alg walks from vertex to vertex. Why??
Understanding the Geometry of LPs

- Today’s lecture: Understanding geometry of LPs

- Focus on inequality constraints, but works with equalities too
  - A few hints along the way

- Provides the foundation for
  - LP formulations
  - Duality
  - Solution methods
  - Conquering the world

The Polyhedron*

- Definition:
  - *Inequality constraints*
  - (Can also contain equalities)

- Visualization

* Sometimes called polytope, nobody can agree on the definition
Extreme Points of a Polyhedron

- Extreme points cannot be represented as a linear combination of two other points in polyhedron
- Examples:

Intuition about extreme points

- An extreme point for a polyhedron in $\mathbb{R}^n$ is:
  - A feasible point
  - The unique intersection of $n$ linearly independent hyperplanes
Active constraints

- Given an LP
  - E.g.,

- An inequality constraint is active at a point $x^*$ if the constraint holds with equality

- BTW. If $x^*$ is a feasible point, then the equality constraints will always be active

Basic solutions

- Consider a polytope:

- Given a set of $n$ linearly independent active constraints

- Basic solution: unique solution for the resulting linear system of linearly independent constraints

- Basic feasible solution: a basic solution that satisfies all constraints

- BTW. In standard form, a basic feasible solution:
  - Satisfies $m$ equality constraints, and
  - $n-m$ inequality constraints $\Rightarrow$
Existence of basic feasible solutions

- Consider a polyhedron P
  - When does a basic feasible solution exist?

- Theorem: If polyhedron is not empty, and there are at least \( n \) linearly independent constraints, then there exists at least one basic feasible solution.

Basic feasible solutions and Extreme points

- Basic feasible solution \( x^* \):
  - Feasible point
  - Unique solution to \( n \) linearly independent

- Extreme point \( x^* \):
  - Cannot be written as a linear combination of other points in polyhedron

- Definitions are quite different
- Theorem: \( x^* \) is a basic feasible solution if and only if \( x^* \) is an extreme point
Vertices of a polyhedron

- A vertex \( x^* \) of a polyhedron \( P \)
  - A point in \( P \) that is uniquely optimal for some objective function \( c \)

- Brings objective function back into the game!

Formally, \( x^* \) is a vertex of \( P \), if
  - \( x^* \) is in \( P \)
  - There exists a cost vector \( c \), such that
    - Cost of \( x^* \) is strictly lower than all other point \( y \) in \( P \)

Vertices, extreme points and basic feasible solutions...

- Extreme points:
  -

- Basic feasible solutions:
  -

- Vertices:
  -

- Very different...

- Theorem:
  - Proof:
    - E.g., \( x^* \) vertex \( \Rightarrow \) \( x^* \) extreme point
      - By definition, if \( x^* \) is a vertex:
        - Assume \( x^* \) is not an extreme point, then there exists \( y, z \) and \( \lambda \)
          - Since \( x^* \) is a vertex:
            - Thus:
Vertices and Optimal Solutions

- **LP problem:**
  - For every vertex \( x^* \), there is a cost vector \( c \)
    - \( x^* \) is optimal for \( c \)
  - What about the other way?
    - For every cost vector (every LP), does there exist a vertex?

Optimality of extreme points

- **LP:**
  - If \( P \)
    - has at least one extreme point, and
    - there exists an optimal solution
    - then there exists an optimal solution which is an extreme point of \( P \)
  - **Proof:**
    - Optimal value \( v \):
      - Set of optimal solutions \( Q \):
        - \( Q \) has extreme points:
          - \( x^* \) is an extreme point of \( Q \), then \( x^* \) is an extreme point of \( P \)

There are more general results in the readings.
What you need to know

- The Polyhedron
- Extreme Points
- Active constraints
- Basic (feasible) solutions
- Vertices of a polyhedron
  - Brings objective function back into the game!
- Vertices, extreme points and basic feasible solutions:
  - Equivalence
- Optimality of extreme points

Convex Sets

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Convex optimization v. Nonlinear optimization

- Linear optimization problems
  - Linear objective, linear constraints
  - Efficient solutions!
- Nonlinear optimization
  - Either nonlinear constraints or objective
  - You will often hear: “problem is nonlinear, no hope to solve it… must use local search, simulated annealing,…”
- Convex optimization
  - Many nonlinear objectives/constraints are convex
  - Efficient solutions
- Real question: “convex v. non-convex?”
  - Not “linear v. nonlinear?”
- Even if problem is non-convex, convexity is useful:
  - Convex relaxations of non-convex problems may have theoretical guarantees
  - Can always obtain convex lower bound to non-convex problem
    - Duality (always) and relaxation (often)
  - Can provide good starting point for local search

Outline to learning about convexity

- General definition of a convex optimization problem:
- Equivalent problem:
- How we’ll learn about these problems:
  1. Convex sets
  2. Convex functions
  3. Important special case: Quadratic programming
  4. Convex optimization problems
  5. Duality and convexity
  6. Algorithms for optimizing convex problems
- Applications will be discussed along the way
- Today: characterizing convex sets and some interesting examples
Definitions of convex sets

- Convex v. Non-convex sets

- Line segment definition:

- Convex combination definition:

- Probabilistic interpretation:
  - If $C \subseteq \mathbb{R}^n$ is convex
  - Define any probability distribution
  - Then

Another view of polyhedra:
Intersection of Halfspaces & Hyperplanes

- Half space:

- Hyperplane:

- Intersection:
Intersection of Convex Sets

- Fundamental Theorem: *Intersection of convex sets is convex*
- What can we say about polyhedra?

Interesting Case: Convex Hull

- A convex combination
  - Convex hull
    - Set of all possible convex combinations

- Interesting fact: "Given set of points in a convex set, their convex hull is contained in this convex set"
General convex hull

- Given some set $C$

- Convex hull of $C$, $\text{conv } C$

- Properties of convex hull:
  - Idempotency:

- Usefulness:

Examples of convex sets we have already seen…

- $\mathbb{R}^n$
- point
- half space
- polyhedron
- line
- line segment
- linear subspace
First non-linear example: Euclidean balls and Ellipsoids

- \( B(x_c, r) \) - ball centered at \( x_c \) centered at \( r \):

- Convexity:

- Ellipsoid:
  - \( (x-x_c)^T \Sigma^{-1} (x-x_c) \leq 1 \)
  - \( \Sigma \) is positive semidefinite

Examples of Norm Balls

- Scaled Euclidian (\( L_2 \))
- \( L_1 \) norm (absolute)
- Mahalanobis
- \( L_\infty \) (max) norm
Norm balls

- Convexity of norm balls
  - Properties of norms:
    - Scaling
    - Triangle inequality

- Norm balls are extremely important in ML

- What about achieving a norm with equality?

Cones

- Set \( C \) is a cone if set is invariant to non-negative scaling

- If the cone is convex, we call it:
  - extremely important in ML (as we'll see)

- A cool cone: The ice cream cone
  - a.k.a. second order cone
Positive semidefinite cone

- Positive semidefinite matrices:
  - Positive semidefinite cone:
    - Alternate definition: Eigenvalues

Convexity:

- Examples in ML:
  - A fundamental convex set
    - Useful in a huge number of applications
    - Basis for very cool approximation algorithms
    - Generalizes pretty many “named” convex optimization problems

Operations that preserve convexity 1: Intersection

- Intersection of convex sets is convex

Examples:
  - Polyhedron
  - Robust linear regression
  - Positive semidefinite cone
Operations that preserve convexity 2: Affine functions

- Affine function: \( f(x) = Ax + b \)
- Set \( S \) is convex
  - Image of \( S \) under \( f \) is convex

- Translation:
- Scaling:
- General affine transformation:

Why is ellipsoid convex?
- \((x-x_c)^T \Sigma^{-1} (x-x_c) \leq 1\)
- \( \Sigma \) is positive semidefinite

Operations that preserve convexity 3: Linear-fractional functions

- Linear fractional functions (affine func. over positive linear func.):
  - Closely related to perspective projections (useful in computer vision)

- Given convex set \( C \), image according to linear fractional function:

- Example:
  - Joint distribution:
  - Conditional distribution:
Separating hyperplane theorem

**Theorem:** Every two non-intersecting convex sets \( C \) and \( D \) have a separating hyperplane:

- Intuition of proof (for special case)
  - Minimum distance between sets:
  - If minimum is achieved in the sets (e.g., both sets closed, and one is bounded), then

Supporting hyperplane

**General definition:** Some set \( C \subseteq \mathbb{R}^n \)

- Point \( x_0 \) on boundary
  - Boundary is the closure of the set minus its interior
- Supporting hyperplane:
  - Geometrically: a tangent at \( x_0 \)
  - Half-space contains \( C \)

**Theorem:** for any non-empty convex set \( C \), and any point \( x_0 \) in the boundary of \( C \), there exists (at least one) supporting hyperplane at \( x_0 \)

- (One) **Converse:** If set \( C \) is closed with non-empty interior, and there is a supporting hyperplane at every boundary point, then \( C \) is convex
What you need to know

- Definitions of convex sets
  - Main examples of convex sets
- Proving a set is convex
- Operations that preserve convexity
  - There are many many many other operations that preserve convexity
    - See book for several more examples
- Separating and supporting hyperplanes