Today…

- Want to solve integer program
  - E.g., vars in \( \{0,1\} \)
- Solve convex relaxation
  - E.g., vars in \([0,1]\)
- If minimizing, relaxed objective lower:
  - Somehow round relaxed solution:
    - Can affect feasibility
    - Can affect costs
- Want integer solution:
- Today: some ideas & strategies for rounding
  - See optional books for many more options & details
Integral basic feasible solutions

- **LP:**

  \[
  \begin{align*}
  & \text{min } c^T x \\
  & Ax \geq b \\
  & 0 \leq x \leq 1
  \end{align*}
  \]

- If all optimal basic feasible solutions are integral, we are done!
  - LP relaxation is optimal!!!

- It is sufficient if all basic feasible solutions are integral
  - When does this happen?
  - A sufficient (but not necessary) condition:

  - Basis \( B \rightarrow \text{rows of } A \text{ in basis } A_B \)

  \[
  x = A_B^{-1} b_B
  \]

  - Integral? 
  - \( \text{trivial to check} \)

  - Invertible? 
  - \( \text{inertial} \)

  - \( \text{easy to kill } A \text{ is integral, but not } A_B \)

  \[
  A_B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A_B^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
  \]

  \[
  A_B^{-1} b_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}
  \]

  \[
  x = A_B^{-1} b_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}
  \]

  - Not integral

  \[
  \text{"key problem" in example } |A_B| > 1
  \]
One sufficient (but not necessary) condition: Totally Unimodular matrix

- Structure of inverse of matrix:
  \[ D^{-1} = \frac{1}{|D|} \begin{pmatrix} c_1 & \cdots & c_n \\ \vdots & \ddots & \vdots \\ c_1 & \cdots & c_n \end{pmatrix} \]
  \[ c_{ij} = (-1)^{i+j} D_{ij} \]
  \[ D = \begin{pmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{m1} & \cdots & D_{mn} \end{pmatrix} \]

- Inverse integral if
  - Determinant: \( |D| \in \{-1, 0, 1\} \)
  - Cofactors: \([-1, 0, 1]\) (can be integral)

Relaxations with Totally Unimodular Matrices

- Defn: Matrix \( A \) is totally unimodular if the determinant of every square submatrix is either -1, 0, or 1

\[ A \times \mathbb{Z}^n \]

- Thm: If an LP has a totally unimodular constraint matrix \( A \), and the vector \( b \) is integral, then all basic feasible solutions are integral
  - Thus LP relaxation is OPT for integer program
How often do you see totally unimodularity?

- Often
  - Bipartite matching
  - Cuts
  - Maximum margin Markov networks

- Not often
  - \( \mathsf{P=NP} \)

- One thing we can agree: it’s usually not easy to spot…

Sufficient conditions for total unimodularity

Matrix A is totally unimodular if

- All entries are -1, 0, or 1
- Each column contains at most two nonzero elements
- Rows of A can be partitioned into two sets \( A_1 \) and \( A_2 \) such that two nonzero entries in a column are
  - in the same set of rows if they have different signs
  - in different sets of rows if they have the same sign

Maximum bipartite matching:

- Two sets of nodes
  - Edges from nodes i in A to j in B have weight \( w_{ij} \)

- Can be solved exactly by LP, even though it is a combinatorial problem

- Related to \( x_{ij} \geq 0 \) and prove that you get some solution
Relaxations and rounding

- What do we do if we don’t get integral solutions?
  - Because \( \emptyset \notin V \) (quickly)

- E.g., set cover problem
  - Ground elements \( v \in V \)
  - Set of sets \( S \subseteq V \)
  - Cost for sets \( C_S \)
  - Find cheapest collection of subsets that covers all elements

- Integer program and relaxation:

- How can we obtain a good integer (rounded) solution?
  - If we set all nonzero \( x_S \) to one, then
  - Smarter way to round?

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