From relaxations to integral solutions

Optimization - 10725
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Today…

- Want to solve integer program
  - E.g., vars in \{0,1\}
- Solve convex relaxation
  - E.g., vars in \([0,1]\)
- If minimizing, relaxed objective lower:

- Want integer solution:
  - Somehow round relaxed solution:
    - Can affect feasibility
    - Can affect costs

- Today: some ideas & strategies for rounding
  - See optional books for many more options & details
Integral basic feasible solutions

- LP:

- If all optimal basic feasible solutions are integral, we are done!
  - LP relaxation is optimal!!!

- It is sufficient if all basic feasible solutions are integral
  - When does this happen?
  - A sufficient (but not necessary) condition:
Integral matrix ➔ Integral inverse?
One sufficient (but not necessary) condition: Totally Unimodular matrix

Structure of inverse of matrix:

- Inverse integral if
  - Determinant:
  - Cofactors:
Relaxations with Totally Unimodular Matrices

- Defn: Matrix $A$ is totally unimodular if the determinant of every square submatrix is either -1, 0, or 1

- Thm: If an LP has a totally unimodular constraint matrix $A$, and the vector $b$ is integral, then all basic feasible solutions are integral

   Thus
How often do you see totally unimodularity?

- Often
  - Bipartite matching
  - Cuts
  - Maximum margin Markov networks

- Not often

One thing we can agree: it’s usually not easy to spot…
Sufficient conditions for total unimodularity

- Matrix A is totally unimodular if:
  - All entries are -1, 0, or 1
  - Each column contains at most two nonzero elements
  - Rows of A can be partitioned into two sets $A_1$ and $A_2$ such that two nonzero entries in a column are:
    - in the same set of rows if they have different signs
    - in different sets of rows if they have the same sign

- Maximum bipartite matching:
  - Two sets of nodes
    - Edges from nodes i in A to j in B have weight $w_{ij}$
Relaxations and rounding

- What do we do if we don’t get integral solutions?
  - E.g., set cover problem
    - Ground elements
    - Set of Sets
    - Cost for sets
    - Find cheapest collection of subsets that covers all elements

- Integer program and relaxation:

- How can we obtain a good integer (rounded) solution?
  - If we set all nonzero $x_s$ to one, then
Consider a special case…

- Suppose each element in at most k sets
- From inequality constraint:

- Rounding strategy:
- Feasibility:
- Cost of rounded solution:
Very simple example of randomized rounding

- Solve set cover relaxation:
  - Randomly pick a collection of subsets $G$
    - For each $S$, add it to $G$ with (independent) probability $x_s$
  - What’s the expected cost of $G$?
    - $I_s$ indicator of whether set $S$ is in $G$
How big can cost get?

- Expected cost is lower than $\text{OPT}_{\text{IP}}$
  - But how big can actual cost get?
  - (a simple bound here, more interesting bounds using more elaborate techniques)

- Markov Inequality: Let $Y$ be a non-negative random variable
  - Then

- In our example:
How many elements do we cover?

- Expected cost of $G$ can be lower than $\text{OPT}_{IP}$
  - Must cover fewer elements

- $I_v$ is indicator of whether element $v$ covered by $G$
- Expected number of elements covered: